

Refutation of politeness, strong politeness, finite witnessability and hence stable-infiniteness

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·, ∘ , ⊗ ; \ Not And;
> Imply, greater than, →, ⇒ , ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ←, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠, ⊕;
% possibility, for one or some, ∃, ∃!, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Notes: for clarity, we usually distribute quantifiers onto each designated variable; and for ordinal arithmetic, the result is implied.

From: Sheng, Y.; et al. (2021). Politeness and stable infiniteness: stronger together. arxiv.org/pdf/2104.11738.pdf

Abstract. We make two contributions to the study of polite combination in satisfiability modulo theories. The first contribution is a separation between politeness and strong politeness, by presenting a polite theory that is not strongly polite. This result shows that proving strong politeness (which is often harder than proving politeness) is sometimes needed in order to use polite combination. The second contribution is an optimization to the polite combination method, obtained by borrowing from the Nelson-Oppen method. In its non-deterministic form, the Nelson-Oppen method is based on guessing arrangements over shared variables. In contrast, polite combination requires an arrangement over *all* variables of the shared sort (not just the shared variables). We show that when using polite combination, if the other theory is stably infinite with respect to a shared sort, only the shared variables of that sort need be considered in arrangements, as in the Nelson-Oppen method. Reasoning about arrangements of variables is exponential in the worst case, so reducing the number of variables that are considered has the potential to improve performance significantly. We show preliminary evidence for this

Related Work: Polite combination is part of a more general effort to replace the stable infiniteness symmetric condition in the Nelson-Oppen approach with a weaker condition. Other examples of this effort include the notions of *shiny* [22], *parametric* [14], and *gentle* [12] theories. Gentle, sinly and polite theories can be combined à la Nelson-Oppen with any arbitrary theory. Shiny theories were introduced by Tinelli and Zarba [22] as a class of mono-sorted theories. Based on the same principles as shininess, politeness is particularly well-suited to deal with theories expressed in many-sorted logic. Polite theories were introduced by Ranise et al. [18] to provide a more effective combination approach compared to parametric and shiny theories, the former requiring solvers to reason about cardinalities and the latter relying on expensive computations of minimal cardinalities of models. Shiny theories were extended to many-sorted signatures in [18], where there is a sufficient condition for their equivalence with polite theories. For the mono-sorted case, a sufficient condition for the equivalence of shiny theories and strongly polite theories was given by Casal and Rasga [8]. In later work [9], the same authors proposed a generalization of shiny theories to many-sorted signatures different from the one in [18], and proved that it is equivalent to strongly polite theories with a decidable quantifier-free fragment. The strong politeness of the theory of algebraic datatypes [5] was proven in [19]. That paper also introduced *additive witnesses*, that provided a sufficient condition for a polite theory to be also strongly polite. In this paper we present a theory that is polite but not strongly polite. In accordance with [19], the witness that we provide for this theory is not additive.

3.2 A Polite Theory that is not Strongly Polite

Let Σ_2 be a signature with two sorts σ_1 and σ_2 and no function or predicate symbols (except $=$). Let $\mathcal{T}_{2,3}$ be the Σ_2-theory from [9], consisting of all Σ_2-structures $\mathcal{A}$ such that either $|\sigma_1^{\mathcal{A}}| = 2 \wedge |\sigma_2^{\mathcal{A}}| \geq \aleph_0$ or $|\sigma_1^{\mathcal{A}}| \geq 3 \wedge |\sigma_2^{\mathcal{A}}| \geq 3$ [9].[7]

$\mathcal{T}_{2,3}$ is polite, but is not strongly polite. Its smoothness is shown by extending any given structure with new elements as much as necessary.

Lemma 1. *$\mathcal{T}_{2,3}$ is smooth w.r.t. $\{\sigma_1, \sigma_2\}$.*

For finite witnessability, consider the function *wit* defined as follows:

$$wit(\phi) := \phi \wedge x_1 = x_1 \wedge x_2 = x_2 \wedge x_3 = x_3 \wedge y_1 = y_1 \wedge y_2 = y_2 \wedge y_3 = y_3 \quad (1)$$

for fresh variables x_1, x_2, and x_3 of sort σ_1 and y_1, y_2, and y_3 of sort σ_2. It can be shown that *wit* is a witness for $\mathcal{T}_{2,3}$ but there is no strong witness for it.

(3.2.1.1)

LET p, q, r, s, t, u, v: φ, x_1, x_2, x_3, y_1, y_2, y_3.

(((p&q)=(q&r))=((r&s)=(s&t)))=((t&u)=((u&v)=v)) ;

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FFFT FFTF FFFT TTFT}1}2}16
FFFT FFTF TTTF FFTF}1} }
FFFT FFTF FFFT TTFT}1} }
FFFT FFTF FFFT TTFT}1}1}
TTTF TTFT FFFT TTFT}1} }
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(3.2.1.2)

Remark 3.2.1.2: Eqs. 3.2.1.2 is *not* tautologous, refuting the definition of the function for finite witnessability, denying a polite theory that is not strongly polite, and replacement of symmetric stable infiniteness.

Conjectures therefrom infer misplaced novelty for shiny, parametric, and gentle terms and dubious utility in computer programming, as the patently obvious in the abstract:

> "Reasoning about arrangements of variables is exponential in the worst case, so reducing the number of variables that are considered has the potential to improve performance significantly."