# Refutation of the Boolean Compactness Theorem in Meth8/VŁ4 Modal Logic

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Abstract

The Boolean Compactness Theorem asserts that a set of Boolean formulas is satisfiable if and only if every finite subset is satisfiable. This paper refutes the theorem within the Meth8/VŁ4 quad-valent modal logic system, which uses truth values F=(0,0), N=(0,1), C=(1,0), and T=(1,1), requiring theorems to be tautologies (all T). By formalizing the theorem as $\%p \equiv \%q$ and evaluating it, alongside test formulas $((p \to q)\&(q \to p))\&((p \to\sim q)\&(q \to\sim p))$ and $((A \to B)\&(B \to A))\&((A \to\sim B)\&(B \to\sim A))$, we demonstrate non-tautologous results, refuting the theorem in Meth8/VŁ4 due to its finite universe constraints.

## 1 Introduction

The Boolean Compactness Theorem is a cornerstone of propositional logic, stating that a set of Boolean formulas $\Gamma$ is satisfiable if every finite subset $\Delta \subseteq \Gamma$ is satisfiable. The Meth8/VŁ4 modal logic system, with quad-valent truth values (F=(0,0), N=(0,1), C=(1,0), T=(1,1)) and a finite universe, offers a framework to test this claim [1]. Theorems in Meth8/VŁ4 must be tautologies (all T). This paper refutes the theorem by showing its formalization and related test formulas are non-tautologous.

## 2 Meth8/VŁ4 Framework

Meth8/VŁ4 is a quad-valent modal logic system with truth values:

- F=(0,0): False
- N=(0,1): Non-contingent
- C=(1,0): Contingent
- T=(1,1): True (tautology)

Connectives (4x4, row-major, F,C,N,T):

- Negation ($\sim$): F→T, T→F, C→N, N→C
- Implication (→): TTTT NTNT CCTT FCNT
- Conjunction (&): FFFF FCFC FFNN FCNT

- Equivalence (≡): TNCF NTFC CFTN FCNT
- Possibility (%): F→C, C→C, N→T, T→T

Variables: propositional ($p = \text{FTFT}^4$, $q = \text{FFTT}^4$), theorem ($A = \text{FCNT}^4$, $B = \text{FFFF CCCC NNNN TT}$
Truth tables have 16 rows. The system rejects infinite sets and supports classical explosion.

# 3 Formalizing the Theorem

The theorem is formalized as:

$$\%p \equiv \%q$$

where $p$ is "$\Gamma$ is satisfiable" and $q$ is "every finite subset $\Delta \subseteq \Gamma$ is satisfiable."

# 4 Evaluation of the Theorem

Using $p = \text{FTFT}^4$, $q = \text{FFTT}^4$:

- $\%p = \text{CTTT}^4$, $\%q = \text{CCCT}^4$
- $\%p \equiv \%q = \text{FCNT}^4$

Result: FCNT FCNT FCNT FCNT (non-tautologous), refuting the theorem as a theorem in Meth8/VŁ4.

# 5 Test Formulas

To further support the refutation, we evaluate: 1. $((p \to q)\&(q \to p))\&((p \to\sim q)\&(q \to\sim p))$:

- $\sim q = \text{TTFF}^4$, $\sim p = \text{TFTF}^4$
- $p \to q = \text{TFTT}^4$, $q \to p = \text{TTFT}^4$
- $p \to\sim q = \text{TTTF}^4$, $q \to\sim p = \text{TTTF}^4$
- $(p \to q)\&(q \to p) = \text{TFFT}^4$
- $(p \to\sim q)\&(q \to\sim p) = \text{TTTF}^4$
- Final: $\text{TFFF}^4$ (non-tautologous)

2. $((A \to B)\&(B \to A))\&((A \to\sim B)\&(B \to\sim A))$:

- $\sim B = \text{TTTT NNNN CCCC FFFF}$, $\sim A = \text{TNCF}^4$
- $A \to B = \text{TNCF TTCC TNTN TTTT}$, $B \to A = \text{TTTT NTNT CCTT FCNT}$
- $A \to\sim B = \text{TTTT TNTN TTCC TNCF}$, $B \to\sim A = \text{TTTT TNTN TTCC TNCF}$
- $(A \to B)\&(B \to A) = \text{TNCF NTFC CFTN FCNT}$
- $(A \to\sim B)\&(B \to\sim A) = \text{TTTT TNTN TTCC TNCF}$
- Final: TNCF NNFF CFCF FFFF (non-tautologous)

# 6 Analysis

The theorem ($\%p \equiv \%q$) and test formulas are non-tautologous, supporting the refutation of the Boolean Compactness Theorem in Meth8/VŁ4. The finite universe and quad-valent logic reject the theorem's infinite set assumptions, as seen in the non-tautologous results of the test formulas, which may model counterexamples to satisfiability equivalences.

# 7 Conclusion

The Boolean Compactness Theorem is refuted in Meth8/VŁ4, as its formalization and supporting test formulas are non-tautologous. This highlights Meth8/VŁ4's unique constraints, offering a new perspective on classical logical principles.

# References


[1] Colin James III, "Meth8/VŁ4 System Description," https://ersatz-systems.com, 2024.